Lecture 14.

This week we will switch to the fantasy genre: cloning, teleportation, etc. A pivotal role in these algorithms is played by EPR pairs or, more generally, entangled states.

Def-n. Let V and W be two finite dimensional vector spaces. A vector $h \in V \otimes W$ is called indecomposable provided h can not be written as $h = v \otimes w$ with $v \in V$ and $w \in W$.

Similarly, let $|\xi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$ be a state vector. Then $|\xi\rangle$ is called entangled provided $|\xi\rangle$ can not be written as $|\xi\rangle = |\psi\rangle \otimes |\varphi\rangle$ with $|\psi\rangle \in \mathcal{H}_1$ and $|\varphi\rangle \in \mathcal{H}_2$.

Example. Let $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{C}^2$ and $|\xi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. We will show that $|\xi\rangle$ is entangled. Indeed, otherwise

$$|\xi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = (\alpha_0|0\rangle + \alpha_1|1\rangle) \otimes (\beta_0|0\rangle + \beta_1|1\rangle) =$$

$$= \alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle$$ meaning

$$\begin{cases} \alpha_0\beta_0 = \frac{1}{\sqrt{2}} \\ \alpha_1\beta_0 = \alpha_0\beta_1 = 0 \\ \alpha_1\beta_1 = \frac{1}{\sqrt{2}}. \end{cases}$$

It is easy to see that the system above has no solutions.

Rmk. The vector $|\xi\rangle$ above is called an EPR pair (after Einstein, Podolsky and Rosen, see their '1935 paper' on Canvas) or Bell state (see his '1964 paper' on Canvas).

No-cloning theorem. There is no operator $U \in \mathcal{U}((\mathbb{C}^2)^{\otimes n})$ such that $U(|\psi\rangle \otimes |0^{n-1}\rangle) = |\psi\rangle \otimes |\psi\rangle \otimes |garbage\rangle$ for all qubits $|\psi\rangle \in \mathbb{C}^2$.

(the $|0^{n-1}\rangle$ are ancillas; $|garbage\rangle \in (\mathbb{C}^2)^{\otimes n-2}$)

Proof. Straightforward: suppose such an operator U exists, then

$$U(|0\rangle \otimes |0^{n-1}\rangle) = |0\rangle \otimes |0\rangle \otimes |garbage_0\rangle$$

$$U(|1\rangle \otimes |0^{n-1}\rangle) = |1\rangle \otimes |1\rangle \otimes |garbage_1\rangle$$

and, as U is linear,

$$U(|+\rangle \otimes |0^{n-1}\rangle) = \frac{1}{\sqrt{2}}\left(U(|0\rangle \otimes |0^{n-1}\rangle) + U(|1\rangle \otimes |0^{n-1}\rangle)\right) =$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle \otimes |0\rangle \otimes |garbage_0\rangle + |1\rangle \otimes |1\rangle \otimes |garbage_1\rangle\right).$$

On the other hand, we would like to have

$$U(|+\rangle \otimes |0^{n-1}\rangle) = |+\rangle \otimes |+\rangle \otimes |garbage_+\rangle.$$

Let's show that the equality

$$(\star)\quad \frac{1}{\sqrt{2}}\left(|0\rangle \otimes |0\rangle \otimes |garbage_0\rangle + |1\rangle \otimes |1\rangle \otimes |garbage_1\rangle\right) = |+\rangle \otimes |+\rangle \otimes |garbage_+\rangle$$

can not hold true.

Indeed, $|+\rangle \otimes |+\rangle \otimes |garbage_+\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \otimes |garbage_+\rangle$.

Measuring the first two qubits on the l.h.s. and r.h.s of $(\star)$, we see that probabilities do not match up. For instance, the probability of getting $|01\rangle$ on the l.h.s is 0, while it is $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$ on the r.h.s.

Rmk. We have actually shown that it is impossible to 'clone' the three state vectors $|0\rangle$, $|1\rangle$ and $|+\rangle$ simultaneously.

Quantum teleportation.

Suppose Alice has a qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, which she would like Bob to have. Let's assume that Alice and Bob have prepared an EPR pair $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, Alice stored the first qubit and Bob stored the second. Moreover, Alice can text Bob two classical bits.

Here is a way to 'teleport' $|\psi\rangle$.

Step 1. Alice has two qubits: $|\psi\rangle$ and the first qubit of the EPR pair. She applies CNOT with $|\psi\rangle$ being the control bit:

$$|\psi\rangle \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = (\alpha|0\rangle + \beta|1\rangle) \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \rightsquigarrow$$

$$\rightsquigarrow \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|110\rangle + \beta|101\rangle).$$

Step 2. Alice measures her second qubit:

$$P\left(\begin{matrix}\text{Alice's second} \\ \text{qubit is } |0\rangle\end{matrix}\right) = \left(\frac{1}{\sqrt{2}}\right)^2 (\underset{\substack{\uparrow \\ \text{from } |0\underline{0}0\rangle}}{|\alpha|^2} + \underset{\substack{\uparrow \\ \text{from } |1\underline{0}1\rangle}}{|\beta|^2}) = \frac{1}{2}.$$

$$P\left(\begin{matrix}\text{Alice's second} \\ \text{qubit is } |1\rangle\end{matrix}\right) = \left(\frac{1}{\sqrt{2}}\right)^2 (\underset{\substack{\uparrow \\ \text{from } |0\underline{1}1\rangle}}{|\alpha|^2} + \underset{\substack{\uparrow \\ \text{from } |1\underline{1}0\rangle}}{|\beta|^2}) = \frac{1}{2} \quad \left(\text{or simply } 1 - \frac{1}{2} = \frac{1}{2}\right)$$

In the first case (2^{nd} qubit is $|0\rangle$) the state collapses to $\frac{1}{\sqrt{2}}(\alpha|00\rangle + \beta|11\rangle)$, while if the second qubit is $|1\rangle$, the state collapses to $\alpha|01\rangle+\beta|00\rangle$, in which case Alice sends Bob the classical bit '1' and he applies a NOT to his state. This way, the resulting state at this point will always be $\frac{1}{\sqrt{2}}(\alpha|00\rangle+\beta|11\rangle)$.

Step 3. Alice applies the Hadamard operator to her qubit: $\frac{1}{\sqrt{2}}(\alpha|00\rangle+\beta|11\rangle) \rightsquigarrow \frac{1}{\sqrt{2}}(\alpha|+0\rangle+\beta|-1\rangle) =$

$$= \frac{1}{\sqrt{2}}(\alpha|00\rangle+\alpha|10\rangle+\beta|01\rangle-\beta|11\rangle).$$

Step 4. Alice measures her qubit:

$$P\left(\begin{matrix}\text{Alice's qubit}\\ \text{collapses to } |0\rangle\end{matrix}\right) = \left(\frac{1}{\sqrt{2}}\right)^2 \left(\underset{\substack{\uparrow\\ \text{from } |00\rangle}}{|\alpha|^2} + \underset{\substack{\uparrow\\ \text{from } |01\rangle}}{|\beta|^2}\right) = \frac{1}{2}$$

$$P\left(\begin{matrix}\text{Alice's qubit}\\ \text{collapses to } |1\rangle\end{matrix}\right) = \frac{1}{2}.$$

1st case: Bob has $\alpha|0\rangle+\beta|1\rangle = |\psi\rangle$ ✓

2nd case: Bob has $\alpha|0\rangle-\beta|1\rangle$. Then Alice sends Bob a 1-bit message '1' (2nd classical bit) and he applies the operator $Z = \begin{pmatrix}1&0\\0&-1\end{pmatrix} \in U_2(\mathbb{C})$ on the qubit to get $Z(\alpha|0\rangle-\beta|1\rangle) = \alpha|0\rangle+\beta|1\rangle = |\psi\rangle$ ✓

Informal slogan: '1 ebit + 2 bits $\geq$ 1 qubit' (Bennet's law)
(ebit $\uparrow$ EPR pair)

Rmk. Alice doesn't have $|\psi\rangle$ anymore, so no-cloning thm was not violated.

Entanglement swapping.

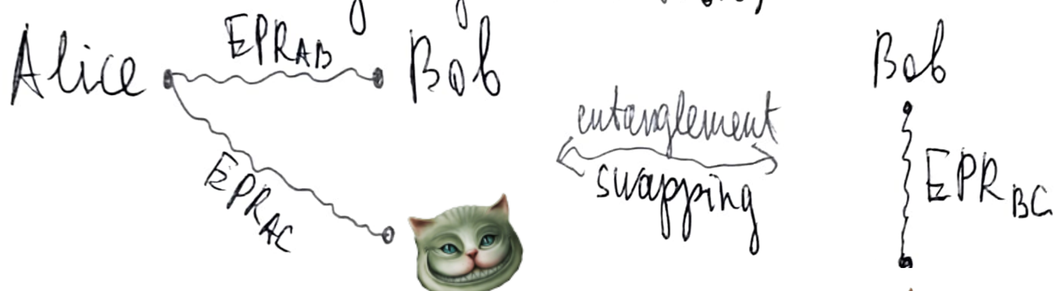
Suppose Alice shares an EPR pair with Bob

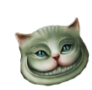
$$EPR_{AB} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

and another EPR pair with Cheshire cat

$$EPR_{AC} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

The entanglement swapping allows to create an entangled pair of qubits between Bob and the Cheshire cat (without them having any interaction):





The procedure is essentially build on Alice's teleportation to Bob her part (qubit) of the EPR_{AC} pair. We give an outline.

① Starting state: $\frac{1}{2}\underbrace{(|00\rangle+|11\rangle)}_{EPR_{AB}} \otimes \underbrace{(|00\rangle+|11\rangle)}_{EPR_{AC}} = \frac{1}{2}(|0000\rangle+|0011\rangle+|1100\rangle+|1111\rangle)$ (the 1st and 3rd qubit belong to Alice).

② Alice applies $CNOT_{13}$ (1st qubit is controlling 3rd) giving the state $\frac{1}{2}(|0000\rangle+|0011\rangle+|1110\rangle+|1101\rangle)$.

③ Alice measures the third qubit producing the state

$\frac{1}{\sqrt{2}}(|0000\rangle+|1111\rangle) \rightsquigarrow$ texts Bob the classical bit 'False' (0).

$\frac{1}{\sqrt{2}}(|0011\rangle+|1110\rangle) \rightarrow$ texts Bob (and ) a 'True' (1) to apply NOT

As a result the state will be $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ in both cases.

(4) Alice applies H on the 1^{st} qubit, the state becomes

$$\frac{1}{\sqrt{2}}(|+00\rangle + |-11\rangle) = \frac{1}{2}(|000\rangle + |100\rangle + |011\rangle - |111\rangle)$$

(5) Alice measures her first qubit:

if $|0\rangle$, the state collapses to $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$;

if $|1\rangle$, the state collapses to $\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle$.

In the first case Alice texts both Bob and the cat 'False'; otherwise she texts one of them 'True' and the other 'False'. The one that receives 'True' applies Z to his qubit.